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THEORY OF ARTIFICIAL STABILIZATION
OF MISSILES AND SPACE VEHICLES
WITH EXPOSITION OF FOUR CONTROL PRINCIPLES

By R. F. Hoelker

George C. Marshall Space Flight Center
Huntsville, Alabama

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LIST OF SYMBOLS

Symbol	
F	Thrust force
X	Axial air force
N	Air force perpendicular to long axis of vehicle
R	Control force perpendicular to long axis of missile, due to motor swiveling or/and due to vane deflections
M	Mass of vehicle
Θ	Moment of inertia of vehicle in yaw or pitch
r	Radius of gyration in yaw or pitch
c_1	Specific aerodynamic restoring torque
c_2	Specific control torque
v	Magnitude of standard velocity of vehicle
w	Wind velocity magnitude
$\ddot{z}$	Linear acceleration of center of gravity of vehicle perpendicular to standard path
$\dot{z}$	Velocity of center of gravity of vehicle perpendicular to standard path
$\ddot{\tau}$	Velocity of center of gravity of vehicle perpendicular to long vehicle axis
a	Local linear acceleration at a vehicle station perpendicular to long vehicle axis
α	Angle of attack
β	Swivel motor deflection or vane deflection
φ	Attitude angle
α_W	Wind angle, between flow and standard path

LIST OF SYMBOLS (CONT'D)

Symbol	
a_o	Attitude displacement gain
a_1	Attitude rate gain
a_2	Attitude acceleration gain
b_o	Angle of attack gain
g_2	Local lateral acceleration gain
C_G	Location of center of gravicy
C_P	Location of aerodynamic center of pressure
C_M	Location of accelerometer
C_H	Location of point of interest (as of human passenger)
A_i	Coefficients of characteristic equation
B_i	Coefficients of path reaction equation
λ	Differential operator
ω	undamped frequency
ζ	Damping ratio

Superscripts

dot	For differentiation with respect to time
prime	For differentiation with respect to angle

Subscripts

qss	For quasi steady state
M	For "measured"
H	For "at place of interest" as location of human passenger
others	Explained at place of occurrence

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THEORY OF ARTIFICIAL STABILIZATION OF MISSILES AND SPACE VEHICLES WITH EXPOSITION OF FOUR CONTROL PRINCIPLES

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SUMMARY

The theory of artificial stabilization has been developed in a sequence of reports, published at this installation (see References 1, 2, and 3). The present paper rederives the theory in a more rigorous fashion and extends the application of the theory beyond the Drift-Minimum-Control mode to three further possible modes of control, that is the Load-Minimum-Control, the Maximum Comfort Control and the Motion Center Control mode. Concern is given in this paper to two instrumental arrangements by which the artificial stabilization can materialize. These are the couplings of (a) the space-referenced gyro with an angle of attack meter and (b) the space-referenced gyro with a missile-mounted accelerometer. Formulas for both applications are derived side by side. The theory is restricted to rigid body assumptions.

INTRODUCTION AND ANALYSIS

A. DERIVATION OF THE PRINCIPAL EQUATIONS

Concern is given in this paper to the rigid body motion of a space vehicle. Hence, the degrees of freedom introduced by the flexibility of the missile or by fluid sloshing are disregarded. Further, the reaction of the control members to the commands of the controlling elements is assumed to occur in the idealized form, that is, actuator lag, response threshold and inertial effects of the swiveling motors are ignored. Further, linearization is applied to the representation of the missile motion itself.

The equations set forth can be understood as representing either a vertical missile flight, with the variables measured then either in the pitch or yaw plane, or a tilted flight, where the variables then reflect the motion perpendicular to the plane of flight. With the terms as defined in the list of symbols (page iii), the equations of motion read:

Lateral Path Motion:

$$\ddot{z} = \frac{F - X}{m}\varphi + \frac{N'}{m}\alpha + \frac{R'}{m}\beta \qquad (1)$$

Angular Motion:

$$\ddot{\varphi} + c_1 \alpha + c_2 \beta = 0 \qquad (2)$$

Angular Relationship:

$$\alpha - \alpha_W = \varphi - \frac{\dot{z}}{v} ; \quad \alpha_W \equiv \frac{w}{v} \qquad (3)$$

For the listing of the control equation, the reader will recall from the Introduction that there are several modes of materializing the method of artificial stabilization, two of which were mentioned. Since it is intended here to analyze both, the method based on angle of attack sensing and that based on missile-fixed accelerometer sensing, the analysis is to be split from here on into two parallel channels. The formulas to be shown, however, are identical with the exception of the terms that concern the sensing elements. Therefore, it proves expedient to treat the two modes of sensing in the following equations jointly, with the understanding that only one of the sensing elements will be used in applications.

The control equation shows the control angle β, which may be a motor swivel angle or the angle of jet vanes, in response to commands from (1st) the attitude φ, as measured by reference to space fixed gyros, (2nd) the rate of change of φ for damping, (3rd) either the angle of attack α, or the local lateral acceleration a_M measured by an accelerometer, sensitive perpendicular to the missile's long axis:

$$\beta = a_0 \varphi + a_1 \dot{\varphi} + \underbrace{b_0 \alpha + g_2 a_M}_{\text{In application, the two terms are mutually exclusive.}} \qquad (4)$$

The factors a_0, a_1, b_0 and g_2, are the gain values of the control system. Alternately setting later b_0 or g_2 to zero, the effect of the non-zeroed control mode can be studied.

It is helpful for further reference to write the control equation in somewhat modified forms. The acceleration measured by the instrument can be broken down in two parts, that is the linear acceleration, $\ddot{\tau}$, of the center of gravity of the missile, perpendicular to its axis, and that part associated with the angular acceleration about the center of gravity. The measured value a_M can be written as

$$a_M = \ddot{\tau} + (C_M - C_G)\, \ddot{\varphi} \qquad (5)$$

with C_G and C_M being the locations of the center of gravity and the instrument, both measured from the missile cross section in which the hinge-point of the control motor is located, taken positive in the forward direction. With this relationship introduced, the control equation reads:

Alternate Form of Control Equation:

$$\beta = a_0 \varphi + a_1 \dot{\varphi} + b_0 \alpha + g_2 \left[\ddot{\tau} + (C_M - C_G) \ddot{\varphi} \right] \quad (6)$$

Sometimes the factors of $\ddot{\varphi}$ are combined to

$$a_2 \ddot{\varphi} = g_2 (C_M - C_G) \ddot{\varphi} \quad (7)$$

which is done to achieve consistency with the gains a_0 and a_1. The treatment of this term, as being separate from the $\ddot{\tau}$ - term, is justified, since by the choice of the measuring station C_M the value of a_2 can be changed independently of the value that g_2 may have. In fact, it is by this choice of two parameters that the accelerometer control mode may be somewhat superior to the angle-of-attack mode, which offers only one parameter (b_0) to vary.

A further substitution may be made, that ties $\ddot{\tau}$ to the angles of the missile motion:

$$\ddot{\tau} = \frac{N'}{m} \alpha + \frac{R'}{m} \beta \quad (8)$$

For convenience, the system of equations in the form used may be repeated here.

$$\ddot{z} = \frac{F - X}{m} \varphi + \frac{N'}{m} \alpha + \frac{R'}{m} \beta \quad (9)$$

$$\alpha - \alpha_W = \varphi - \frac{\dot{z}}{V} \quad (10)$$

$$\ddot{\varphi} + c_1 \alpha + c_2 \beta = 0 \quad (11)$$

$$\beta = a_0 \varphi + a_1 \dot{\varphi} + a_2 \ddot{\varphi} + b_0 \alpha + g_2 \left(\frac{N'}{m} \alpha + \frac{R'}{m} \beta\right) \quad (12)$$

By substituting for α and β in equations (9) and (11) the system may be reduced to the following two equations.

Rotary Motion:

$$\ddot{\varphi}\left(1 + a_2 c_2 - g_2 \frac{R'}{m}\right) + \dot{\varphi} a_1 c_2 + \tag{13}$$

$$+ \varphi \left[c_1 + c_2 (a_o + b_o) + g_2 \left(c_2 \frac{N'}{m} - c_1 \frac{R'}{m}\right)\right]$$

$$= \left(\frac{\dot{z}}{v} - \frac{w}{v}\right) \left[c_1 + c_2 b_o + g_2 \left(c_2 \frac{N'}{m} - c_1 \frac{R'}{m}\right)\right]$$

and the equation

$$\ddot{z}\left(1 - g_2 \frac{R'}{m}\right) + \left(\frac{\dot{z}}{v} - \frac{w}{v}\right)\left(\frac{N'}{m} + b_o \frac{R'}{m}\right) =$$

$$= \ddot{\varphi} a_2 \frac{R'}{m} + \dot{\varphi} a_1 \frac{R'}{m} +$$

$$+ \varphi \left[\left(a_o + b_o - g_2 \frac{F - X}{m}\right) \frac{R'}{m} + \frac{F - X + N'}{m}\right] \tag{14}$$

The left-hand member of Equation (13) is usually considered describing the dynamical behavior of the rotary loop (i.e., the motion of the vehicle about its center of gravity.) We will soon check on the validity of this assumption by studying the characteristic equation.

To arrive at an equation that is informative for the reaction of the path motion to the rotary system, we substitute in equation (14) the term $(\dot{z} - w)/v$ by using equation (13). This leads to the following equation:

Path-Reaction:

$$\ddot{z} = \ddot{\varphi}\,\frac{-\frac{N'}{m} - b_0\frac{R'}{m} - a_2\left(c_2\frac{N'}{m} - c_1\frac{R'}{m}\right)}{c_1 + c_2 b_0 + g_2\left(c_2\frac{N'}{m} - c_1\frac{R'}{m}\right)} + \dot{\varphi}\,\frac{-a_1\left(c_2\frac{N'}{m} - c_1\frac{R'}{m}\right)}{c_1 + c_2 b_0 + g_2\left(c_2\frac{N'}{m} - c_1\frac{R'}{m}\right)} \tag{15}$$

$$+ \varphi\,\frac{1}{c_1 + c_2 b_0 + g_2\left(c_2\frac{N'}{m} - c_1\frac{R'}{m}\right)}\left\{\left[-a_0\left(c_2\frac{N'}{m} - c_1\frac{R'}{m}\right) + \frac{F - X}{m}\left[c_1 + c_2 b_0 + g_2\left(c_2\frac{N'}{m} - c_1\frac{R'}{m}\right)\right]\right]\right\}$$

which is later used in the abbreviated form as

$$\ddot{z} = \ddot{\varphi}\, B_2 + \dot{\varphi}\, B_1 + \varphi\, B_0.$$

Before proceeding in the analysis, we list in the following the quasi-steady state solutions for α, β and φ that result, if first and higher derivatives of these variables are assumed to be zero. A quasi-steady state is approached, if the rotary loop is well damped and during a cycle of this loop's motion the change of wind and of the lateral linear velocity $\dot{z}$ is negligible. Then e.g. we can directly derive from equation (13) the following relationship:

$$\varphi_{qss} = \left(\frac{\dot{z}}{v} - \frac{w}{v}\right) \frac{c_1 + c_2\, b_0 + g_2 \left(c_2 \frac{N'}{m} - c_1 \frac{R'}{m}\right)}{c_1 + c_2\,(a_0 + b_0) + g_2 \left(c_2 \frac{N'}{m} - c_1 \frac{R'}{m}\right)} \tag{16}$$

By using equations (10), (11) and (12), we find the following quasi-steady states for α and β.

$$\alpha_{qss} = \left(\frac{\dot{z}}{v} - \frac{w}{v}\right) \frac{-\,a_0\, c_2}{c_1 + c_2\,(a_0 + b_0) + g_2 \left(c_2 \frac{N'}{m} - c_1 \frac{R'}{m}\right)} \tag{17}$$

$$\beta_{qss} = \left(\frac{\dot{z}}{v} - \frac{w}{v}\right) \frac{a_0\, c_1}{c_1 + c_2\,(a_0 + b_0) + g_2 \left(c_2 \frac{N'}{m} - c_1 \frac{R'}{m}\right)} \tag{18}$$

The developed equations (13), (15) and (16) to (18) will be discussed in more detail, after in the next chapter the characteristic equation is derived.

B. THE CHARACTERISTIC EQUATION

Introducing the operator λ and higher powers of it for representing the first and higher derivatives of variables, the homogeneous part of the system (9) to (12) can be written in the following form:

(19)

$\dot{z}$	α	β	φ
$-\lambda$	$\frac{N'}{m}$	$\frac{R'}{m}$	$\frac{F - X}{m}$
0	c_1	c_2	λ^2
$\frac{1}{v}$	1	0	-1
0	$b_0 + g_2 \frac{N'}{m}$	$-1 + g_2 \frac{R'}{m}$	$a_0 + a_1 \lambda + a_2 \lambda^2$

From this, the following characteristic equation is developed.

$$A_3 \lambda^3 + A_2 \lambda^2 + A_1 \lambda + A_0 = 0 \quad \text{with}$$

$$A_3 = a_2 c_2 + 1 - g_2 \frac{R'}{m}$$

$$A_2 = a_1 c_2 + \frac{1}{v} \left(\frac{N'}{m} + b_0 \frac{R'}{m}\right) + \frac{a_2}{v} \left(c_2 \frac{N'}{m} - c_1 \frac{R'}{m}\right) \tag{20}$$

$$A_1 = a_0 c_2 + b_0 c_2 + c_1 + g_2 \left(c_2 \frac{N'}{m} - c_1 \frac{R'}{m}\right) + \frac{a_1}{v} \left(c_2 \frac{N'}{m} - c_1 \frac{R'}{m}\right)$$

$$A_0 = - \left[c_1 + c_2 b_0 + g_2 \left(c_2 \frac{N'}{m} - c_1 \frac{R'}{m}\right)\right] \frac{F - X}{vm} + \frac{a_0}{v} \left(c_2 \frac{N'}{m} - c_1 \frac{R'}{m}\right)$$

When comparing this equation with the left-hand terms of equation (13), it will be noticed that in (13) all terms are neglected that are enclosed in the dashed marking.

The neglected terms are identified as originating in the "path-motion-equation" by showing the velocity v in the denominator (see equation (19) for this).

The system, in general, has three roots. Though principally all three participate in both the path motion and the rotary motion, one of the roots can frequently be thought of as describing the path motion, while the remaining two are thought representing the rotary motion. The association of motions with roots is approximately valid, if the roots are well separated in the complex plane and simultaneously the "path-root" is approaching zero. If this is the case, then the path root is approximated well by the ratio A_0/A_1, where A_0 and A_1 are the last two terms of the characteristic equation. Following this line of reasoning, A_0 equal to zero represents a path motion coordinated to a zero-root, which is a motion essentially determined by the initial conditions and constant (in velocity) with time. This is discussed in more detail in the next chapter.

C. THE DRIFT-MINIMUM-PRINCIPLE

By an appropriate choice of the gain values a_0, b_0 (g_2 respectively), the zero-order term A_0 in the characteristic equation (20) can be made identically zero. As we have discussed in the last chapter, this results in one root of the system to be zero, which root we associated to the lateral path motion $\dot{z}$. The physical significance of A_0 being zero may more clearly be apprehended, if it is noted that the term A_0 is recognized also in the "Path-Reaction" equation (15) representing the coefficient of φ with the exception of the factor v. The structure of (15) is

$$\ddot{z} = B_2 \ddot{\varphi} + B_1 \dot{\varphi} + B_0 \varphi \qquad (21)$$

If $A_0 = 0$, then $B_0 = 0$, which latter condition is identical to stating that for any steady state condition of φ, that is, if $\dot{\varphi}$ and $\ddot{\varphi}$ both are equal to zero, there is no lateral acceleration ($\ddot{z}$) on the center of gravity of the vehicle. Since equation (15) was derived without making any condition on the wind magnitude, the zero lateral acceleration condition is also independent of the wind magnitude. This condition was in the past designated the condition of the "Drift-Minimum-Principle." Since in transient motion, where $\dot{\varphi}$ and $\ddot{\varphi}$ are of finite values, also transient non-zero values for $\ddot{z}$ are to be expected, the claim for zero-drift cannot be made. The condition may be written down explicitly:

Drift-Minimum-Condition

Lateral linear accelerations from steady-state angles will be zero, if the gains a_0 and b_0 (g_2 respectively) are such that

$$\frac{\frac{c_1}{c_2} + b_0 + g_2\left(\frac{F'}{m} - \frac{c_1}{c_2}\frac{R'}{m}\right)}{a_0} = \frac{N' - \frac{c_1}{c_2} R'}{F - X} \qquad (22)$$

The condition requires the two gain values a_o and b_o (or a_o and g_2) to satisfy only a linear relationship. A second condition may be formulated to determine their values.

The straight line in the (a_o, b_o)-plane, that represents the linear relationship, divides the (a_o, b_o)-plane into two areas. Depending on the area in which a combination of a_o, b_o is situated, the vehicle will fly into the wind or drift with the wind. For the first case, the root that is coordinated to the path-motion is positive and therefore unstable, for the second case, the path root is stable and indicative for the rate with which the vehicle's lateral velocity approaches the wind velocity.

The angular attitudes in the Drift-Minimum-Case are true steady states and follow (16) to (18) with $\dot{z}$ being constant and small compared with the wind velocity w. The drift-minimum case may also be defined as that case where, by the control mode, an attitude and throw-angle combination is enforced that leads to the cancellation of the sum of all force components perpendicular to the nominal flight plane.

D. THE LOAD-MINIMUM-PRINCIPLE

From the equations (17) and (18), it is evident that steady-state angles of attack and steady state throw angles are reduced by reducing the gain a_o. An extreme case is obviously given by having a_o equal to zero. Since the two steady state angles here are zero, this condition may be called the load-minimum-condition. In this situation, the missile axis aligns in the flow direction like a weathercock stable missile without special attitude control. The attitude angle φ, therefore, can assume large values. A true steady-state would be reached only with the flight direction being parallel and opposite to the wind direction. Though, with respect to the nominal flight plane, the path-motion is unstable, short time application of this load-minimum-control mode for the benefit of load reduction and throw angle reduction is not objectionable, provided a temporary deviation from the nominal flight plane is permissible.

It may be cautioned that the effect of load reduction is shown here only for quasi-steady states. No deductions can be made from equations (17) and (18) as to the transient dynamical behavior of the angles.

E. THE MAXIMUM COMFORT PRINCIPLE

A third principle may briefly be listed, that can be effected by means of particular gain value choice. It will soon be apparent why this is called the "maximum comfort principle."

The combination of a linear acceleration ($\ddot{z}$) of the center of gravity and of an angular acceleration ($\ddot{\varphi}$) about the center of gravity determines for the points along the missile axis a linear distribution of the local point acceleration, as for instance, measured by local accelerometers, mounted missile-fixed and sensitive perpendicularly to the long missile axis. The local normal acceleration a_H may be expressed as

$$a_H = \ddot{\tau} + (C_H - C_G)\,\ddot{\varphi} \tag{23}$$

with $C_H - C_G$ = distance of point of interest (C_H) from center of gravity (C_G) positive forward of C_G. (The letter "H" taken to indicate e. g., the location of a "human passenger.")

It is conceivable that it is desired for a certain point of the vehicle to be free of normal acceleration as much as possible. This desire is expressed by the requirement that

$$a_H = 0, \quad \text{or} \quad \ddot{\tau} + (C_H - C_G)\,\ddot{\varphi} = 0 \tag{24}$$

Making use of (8) and (9), equation (24) can be written as

$$\ddot{z} = \frac{F - X}{m}\,\varphi - (C_H - C_G)\,\ddot{\varphi} \tag{25}$$

which, combined with equation (15) gives the following criterion:

Maximum-Comfort-Principle

$$0 = \ddot{\varphi}\left[(c_H - c_G) - \frac{\frac{N'}{m} + b_o \frac{R'}{m} + a_2 \left(c_2 \frac{N'}{m} - c_1 \frac{R'}{m}\right)}{c_1 + c_2 b_o + g_2 \left(c_2 \frac{N'}{m} - c_1 \frac{R'}{m}\right)}\right]$$

$$+ \dot{\varphi} \frac{- a_1 \left(c_2 \frac{N'}{m} - c_1 \frac{R'}{m}\right)}{c_1 + c_2 b_o + g_2 \left(c_2 \frac{N'}{m} - c_1 \frac{R'}{m}\right)} \qquad (26)$$

$$+ \varphi \frac{- a_o \left(c_2 \frac{N'}{m} - c_1 \frac{R'}{m}\right)}{c_1 + c_2 b_o + g_2 \left(c_2 \frac{N'}{m} - c_1 \frac{R'}{m}\right)}$$

A full realization of this criterion would be achieved only by having $a_o = a_1 = 0$ and choosing b_o (or a_2, g_2) so that the factor to $\ddot{\varphi}$ is identically zero. Since cutting of a_o and a_1 to zero will, in general, not be feasible, a partial fulfillment of equation (26) only seems to be possible by having the factor to $\ddot{\varphi}$ identically zero. This is expressed by the condition:

$$\left[c_1 (c_H - c_G) - \frac{N'}{m}\right] + b_o \left[c_2 (c_H - c_G) - \frac{R'}{m}\right] + \qquad (27)$$

$$+ \left(c_2 \frac{N'}{m} - c_1 \frac{R'}{m}\right) \left[g_2 (c_H - c_G) - a_2\right] = 0$$

The true realization of criterion (26) could be of interest e.g., for maintaining a fluid level always perpendicular to the long axis of the vehicle.

F. PRINCIPLE OF MOTION CENTER CONTROL

As last case, a modification to the Maximum Comfort Principle may be mentioned. This is concerned with the effort of controlling the point of minimum motion, which is the point on the long axis of vehicle, at which the rotation about the center of gravity and the linear motion of the center of gravity perpendicular to the nominal flight plane cancel each other. In contrast to the former case, here the acceleration $\ddot{z}$, perpendicular to the nominal path, instead of $\ddot{\tau}$, which is normal to the missile axis, is to be considered. The acceleration of a vehicle point $(C_H - C_G)$, perpendicular to the nominal path, is expressed by

$$\ddot{z}_H = \ddot{z} + (C_H - C_G)\,\ddot{\varphi}, \quad \text{which for } \ddot{z}_H = 0, \text{ leads to}$$

$$\ddot{z} = -\,(C_H - C_G)\,\ddot{\varphi}. \tag{28}$$

Feeding this requirement into the path-reaction equation (15) results in the following criterion:

Motion Center Control

$$0 = \ddot{\varphi}\left[(C_H - C_G) - \frac{\dfrac{N'}{m} + b_0\dfrac{R'}{m} + a_2\left(c_2\dfrac{N'}{m} - c_1\dfrac{R'}{m}\right)}{c_1 + c_2 b_0 + g_2\left(c_2\dfrac{N'}{m} - c_1\dfrac{R'}{m}\right)}\right] + \dot{\varphi}\,\frac{-\,a_1\left(c_2\dfrac{N'}{m} - c_1\dfrac{R'}{m}\right)}{c_1 + c_2 b_0 + g_2\left(c_2\dfrac{N'}{m} - c_1\dfrac{R'}{m}\right)}$$
$$+\,\varphi\,\frac{-\,a_0\left(c_2\dfrac{N'}{m} - c_1\dfrac{R'}{m}\right) + \dfrac{F - X}{m}\left[c_1 + c_2 b_0 + g_2\left(c_2\dfrac{N'}{m} - c_1\dfrac{R'}{m}\right)\right]}{c_1 + c_2 b_0 + g_2\left(c_2\dfrac{N'}{m} - c_1\dfrac{R'}{m}\right)} \tag{29}$$

It is interesting to note by comparison of (29) with (26) that the factors of $\dot{\varphi}$ and $\dot{\varphi}$ are identical in the two equations, that, however, the factors of φ in (26) and (29) reflect exactly the two earlier discussed principles of load minimum and drift minimum control.

The physical significance of the Motion Center Control Mode is that at the location of the Motion Center, e. g., a fluid layer remains parallel to the horizontal (in vertical flight) and the layers above the motion center experience an acceleration opposite in sign to those below the motion center. (This assumes no interaction between layers.)

If we ignore the term with $\dot{\varphi}$, the inter-relationship between the four discussed principles of control may then be summarized in the following way:

1. The minimum load principle calls for $a_0 = 0$.

2. The minimum drift principle requires that a_0 and b_0 (or g_2) satisfy equation (22) which reads:

$$\frac{c_1}{c_2} + b_0 + g_2 \left(\frac{N'}{m} - \frac{c_1}{c_2}\frac{R'}{m}\right) = a_0 \frac{N' - \frac{c_1}{c_2} R'}{F - X} \qquad (22)$$

3. The maximum comfort principle (or vehicle related control of fluid levels) calls for a combined satisfaction of (1st) the load minimum principle and (2nd) the equation (27) which reads:

$$\left[c_1 (C_H - C_G) - \frac{N'}{m}\right] + b_0 \left[c_2 (C_H - C_G) - \frac{R'}{m}\right] + \\ + \left[g_2 (C_H - C_G) - a_2\right] \left(c_2 \frac{N'}{m} - c_1 \frac{R'}{m}\right) = 0 \qquad (27)$$

4. The motion center control requires satisfying also the just listed equation (27), but now in combination with the drift minimum principle, (equation 22).

G. LIMITATIONS IN SATISFYING SIMULTANEOUSLY SEVERAL PRINCIPLES

The possibility of satisfying several requirements simultaneously depends on the number of open parameters that are available. The variable a_1 is determining primarily the degree of damping of the rotary motion, as shown in the $\dot{\varphi}$-term of equation (13), though the rooting of equation (20) shows that the damping is also slightly dependent on other parameters. The damping characteristic is usually allowed to vary in large limits (as from 40% to 140% of critical damping), but it is not allowed to be zero. This excludes the term of $\dot{\varphi}$ in (15) to vanish identically.

A demand on a_0 and b_0 (or g_2) is imposed from considerations of the rigid body frequency. An approximate expression for the frequency of the rotary motion is derived from the coefficient of φ in equation (13) with

$$\omega^2 = \frac{c_1 + c_2 (a_0 + b_0) + g_2 \left(c_2 \frac{N'}{m} - c_1 \frac{R'}{m}\right)}{1 + a_2 c_2 - g_2 \frac{R'}{m}} \qquad (30)$$

with ω being the undamped frequency (in rad per sec).

It is obvious that two requirements can be met, since two gains are at our disposition, a_0 and b_0 (g_2, respectively). Thus, besides the frequency, we can e. g., choose to go to drift minimum or load minimum or any path reaction mode between these two.

Further, in the particular case of utilizing an accelerometer, the location of this instrument along the long vehicle axis can be chosen arbitrarily (in bounds) which provides us a third variable. This location is reflected in the magnitude of the gain a_2. By virtue of this variable, the accelerometer-based vehicle control arrangement is superior to the angle of attack-based arrangement. (It may be recalled here that concern is given here only to the rigid body mode.)

H. COMPILATION OF FORMULAS IN REDUCED FORM

The formulas listed in the previous chapters adhere to the symbols that were introduced in the original force and moment equations. By taking account of the relationships that exist between forces and moments, the formulas will be simplified, which also facilitates better understanding for the relative significance of the physical parameters.

Distances along the long vehicle axis are referred to the cross-plane of the motor swivel points as origin, this location (if needed) being marked by C_C.

C_C - location of "control center", that is, hinge point of motor or of control vanes.

C_G = location of center of gravity (for pitch or yaw motion).

C_P = location of aerodynamic center of pressure of vehicle

C_M = location of accelerometer, measuring perpendicularly to the long missile axis

C_H = location of point of interest, as that of the human pilot

The direction forward of the control center, C_C, is assigned the positive sign. Since C_C is assigned the origin, the relations hold:

$$C_G - C_C = C_G$$

$$C_P - C_C = C_P, \text{ etc.}$$

Besides these signed linear dimensions, the only other linear quantity that is introduced is the radius of gyration r, which is positive and defined by

$$\Theta = m\, r^2$$

where Θ is the moment of inertia in pitch or yaw and m is the mass of the vehicle.

Then the aerodynamic restoring torque is given by

$$\frac{\partial N}{\partial \alpha}\, \alpha\, (C_G - C_P) = N'\, \alpha\, (C_G - C_P)$$

from which follows for c_1:

$$c_1 = \frac{N'\,(C_G - C_P)}{m\, r^2}$$

The control torque (from motor or vane-deflections) is given by

$$\frac{\partial R}{\partial \beta}\, \beta\, C_G = R'\, \beta\, C_G$$

from which follows:

$$c_2 = \frac{R' \, C_G}{m \, r^2}$$

Further, the ratio c_1/c_2 is expressed by:

$$\frac{c_1}{c_2} = \frac{N' \, (C_G - C_P)}{R' \, C_G}$$

The following terms that frequently occur are listed for convenience, using the just derived relationships:

$$c_2 \, N' - c_1 \, R' = \frac{N' \, R'}{m \, r^2} \left[C_G - (C_G - C_P) \right] = \frac{N' \, R'}{m \, r^2} \, C_P$$

$$\frac{c_1}{c_2} \, R' = N' \, \frac{C_G - C_P}{C_G} = N' \, (1 - \frac{C_P}{C_G})$$

$$N' - \frac{c_1}{c_2} \, R' = N' \, \frac{C_P}{C_G}$$

Using these relationships, we reformulate the most important of the formulas derived in the former chapters.

Rotary Motion (Equation 13)

$$\ddot{\varphi}\left(1 + a_2 \frac{R' C_G}{m r^2} - g_2 \frac{R'}{m}\right) + \dot{\varphi}\, a_1 \frac{R' C_G}{m r^2} + \tag{13-A}$$

$$+ \varphi \left[\frac{N' (C_G - C_P)}{m r^2} + (a_o + b_o) \frac{R' C_G}{m r^2} + \frac{g_2}{m} \frac{N' R'}{m r^2} C_P\right]$$

$$= \left(\frac{\dot{z}}{v} - \frac{w}{v}\right) \left[\frac{N' (C_G - C_P)}{m r^2} + b_o \frac{R' C_G}{m r^2} + \frac{g_2}{m} \frac{N' R'}{m r^2} C_P\right]$$

Undamped frequency of Rotary Motion, if sufficiently separated from the path motion:

(a) for angle of attack control:

$$\omega^2 = \frac{1}{m r^2} \left[N' (C_G - C_P) + R' C_G (a_o + b_o)\right] \tag{30-A}$$

(b) for accelerometer control:

$$\omega^2 = \frac{1}{m r^2} \frac{N' (C_G - C_P) + R' \left(a_o C_G + g_2 \frac{N'}{m} C_P\right)}{1 + g_2 \frac{R'}{m} \left(\frac{a_2 C_G}{g_2 r^2} - 1\right)} \tag{30-B}$$

Using equation (7), the last formula can be written by the use of C_M which is the location of the measuring accelerometer, instead of using a_2:

$$\omega^2 = \frac{1}{m r^2} \frac{N' (C_G - C_P) + R' \left(a_o C_G + g_2 \frac{N'}{m} C_P\right)}{1 + g_2 \frac{R'}{m} \left(\frac{C_G (C_M - C_G)}{r^2} - 1\right)} \tag{30-C}$$

Ratio ζ of damping to critical damping:

(a) for angle of attack control:

$$\zeta = \frac{a_1 \frac{R' C_G}{m r^2}}{2\omega} \tag{31-A}$$

or

$$\zeta = \frac{a_1 R' C_G}{2\sqrt{m r^2 \left[N' (C_G - C_P) + (a_0 + b_0) R' C_G\right]}} \tag{31-B}$$

(b) for accelerometer control:

$$\zeta = \frac{a_1 R' C_G}{2\sqrt{1 + g_2 \frac{R'}{m} \left[\frac{1}{r^2} C_G (C_M - C_G) - 1\right]}}$$

$$\cdot \frac{1}{\sqrt{m r^2 \left[N' (C_G - C_P) + R' (a_0 C_G + g_2 \frac{N'}{m} C_P)\right]}} \tag{31-C}$$

Path Reaction (equation 15)

(a) for angle of attack control:

$$\ddot{z} = \ddot{\varphi} r^2 \frac{N' + b_0 R'}{N' (C_P - C_G) - b_0 R' C_G} + \dot{\varphi} \frac{a_1 \frac{R'}{m} N' C_P}{N' (C_P - C_G) - b_0 R' C_G} +$$

$$+ \varphi \left[\frac{a_0 \frac{R'}{m} N' C_P}{N' (C_P - C_G) - b_0 R' C_G} + \frac{F - X}{m}\right] \tag{15-A}$$

(b) for accelerometer control:

$$\ddot{z} = \ddot{\varphi}\,\frac{r^2 + a_2 \dfrac{R'}{m} C_P}{C_P - C_G - g_2 \dfrac{R'}{m} C_P} + \tag{15-B}$$

$$+ \dot{\varphi}\,\frac{a_1 \dfrac{R'}{m} C_P}{C_P - C_G - g_2 \dfrac{R'}{m} C_P} +$$

$$+ \quad \varphi \left[\frac{a_0 \dfrac{R'}{m} C_P}{C_P - C_G - g_2 \dfrac{R'}{m} C_P} + \frac{F - X}{m} \right]$$

Again in the last form, a_2 can be replaced by $g_2\ (C_M - C_G)$, to show directly the influence of the location of the measuring instrument:

$$\ddot{z} = \ddot{\varphi}\,\frac{r^2 + g_2 \dfrac{R'}{m} C_P\,(C_M - C_G)}{C_P - C_G - g_2 \dfrac{R'}{m} C_P} + \dot{\varphi}\,\frac{a_1 \dfrac{R'}{m} C_P}{C_P - C_G - g_2 \dfrac{R'}{m} C_P} +$$

$$+ \quad \varphi \left[\frac{a_0 \dfrac{R'}{m} C_P}{C_P - C_G - g_2 \dfrac{R'}{m} C_P} + \frac{F - X}{m} \right] \tag{15-C}$$

Quasi-Steady States (equations 16 thru 18)

(a) for angle of attack control:

$$\varphi_{qss} = \frac{-\,N'\,(C_G - C_P) - R'\,C_G\,b_o}{N'\,(C_G - C_P) + R'\,C_G\,(a_o + b_o)}\;\frac{w - \dot{z}}{v} \qquad (16\text{-}A)$$

$$\alpha_{qss} = \frac{a_o\,R'\,C_G}{N'\,(C_G - C_P) + R'\,C_G\,(a_o + b_o)}\;\frac{w - \dot{z}}{v} \qquad (17\text{-}A)$$

$$\beta_{qss} = \frac{-\,a_o\,N'\,(C_G - C_P)}{N'\,(C_G - C_P) + R'\,C_G\,(a_o + b_o)}\;\frac{w - \dot{z}}{v} \qquad (18\text{-}A)$$

(b) for accelerometer control:

$$\varphi_{qss} = \frac{N'\left[C_P - C_G - g_z\,\frac{R'}{m}\,C_P\right]}{a_o\,R'\,C_G - N'\,(C_P - C_G - g_z\,\frac{R'}{m}\,C_P)}\;\frac{w - \dot{z}}{v} \qquad (16\text{-}B)$$

$$\alpha_{qss} = \frac{a_o\,R'\,C_G}{a_o\,R'\,C_G - N'\,(C_P - C_G - g_z\,\frac{R'}{m}\,C_P)}\;\frac{w - \dot{z}}{v} \qquad (17\text{-}B)$$

$$\beta_{qss} = \frac{-\,a_o\,N'\,(C_G - C_P)}{a_o\,R'\,C_G - N'\,(C_P - C_G - g_z\,\frac{R'}{m}\,C_P)}\;\frac{w - \dot{z}}{v} \qquad (18\text{-}B)$$

The Drift-Minimum-Condition (equation 22)

(a) for angle of attack control:

$$a_o \frac{C_P}{F - X} - b_o \frac{C_G}{N'} = \frac{C_G - C_P}{R'} \qquad (22\text{-}A)$$

(b) for accelerometer control:

$$a_o \frac{C_P}{F - X} - g_2 \frac{C_P}{m} = \frac{C_G - C_P}{R'} \qquad (22\text{-}B)$$

The supplementary condition for both the Maximum Comfort Principle and Motion Center Control Mode.(equation 27)

(a) for angle of attack control:

$$b_o = - \frac{N'}{R'} \frac{(C_G - C_P)(C_H - C_G) - r^2}{C_G (C_H - C_G) - r^2} \qquad (27\text{-}A)$$

(b) for accelerometer control:

$$g_2 (C_H - C_G) - a_2 = \frac{m}{R'} \frac{(C_G - C_P)(C_H - C_G) - r^2}{C_P} \qquad (27\text{-}B)$$

When substituting for a_2 according to equation (7):

$$g_2 = + \frac{m}{R'} \frac{(C_G - C_P)(C_H - C_G) - r^2}{(C_M - C_H) C_P} \qquad (27\text{-}C)$$

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